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Versal deformations of elements of real classical Lie algebras†

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Abstract. Versal deformations of normal forms of elements of real classical Lie algebras $o(p, q)$, $sp(2n, \mathbb{R})$, $o^*(2n)$, $sp(2p, 2q)$ and $u(p, q)$ are computed. Examples involving the Lie algebras $o(p, q)$, $p+q \leq 6$, are considered.

1. Introduction

In this paper we compute versal deformations of elements of semi-simple real Lie algebras. These deformations can be understood as an efficient and exhaustive description of 'perturbations' of elements within a Lie algebra, considered up to an equivalence transformation gMg^{-1} by elements g of the corresponding Lie group. To be specific consider the Lie algebra o(4, 2) of the conformal group of space–time. Its elements are linear combinations of operators of rotations, Lorentz boosts, translations in space and time, proper conformal transformations and a dilation (see appendix). Given one of these operators, for instance the Lorentz boost K_2, how can one describe all others which are 'close' to the given one? An obvious answer is to add to K_2 a linear combination of the 15 generators with small coefficients. Indeed, any given element M of o(4, 2) which differs a little from K_2 must be among the linear combinations. However, among these there will be many elements conjugate to M under the action of the group O(4, 2) of inner automorphisms of o(4, 2). In order to avoid such a redundancy and to have each orbit of o(4, 2) elements found in the vicinity of K_2 represented exactly once, one has to determine a minimal versal deformation of K_2. A particularly simple versal deformation of it is

$$K_2+\lambda_1K_2+\lambda_2P_1+\lambda_3C_1+\lambda_4P_3+\lambda_5C_3+\lambda_6L_2+\lambda_7D \qquad (1.1)$$

where λ_i are 'independent' small real parameters. Thus every 'perturbation' of K_2 or an element of o(4, 2) conjugate to it is conjugate to (1.1) for some values of the parameters λ_i. The number of parameters involved gives information about the complexity of the orbit. The more singular an element is, the more parameters are involved.

Our work originated from a paper by Arnold (1971) in which he remarks that both the Jordan normal form M_J of a complex matrix M and the reducing matrix g, where

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$M_J = gMg^{-1}$, depend discontinuously on the elements of the original matrix M. So if a matrix has values which are known only approximately, it is unwise to reduce it to its Jordan form. Also, if one has a family of matrices depending smoothly on parameters, each individual matrix can be reduced to a Jordan normal form but in such an operation the smoothness and even the continuity relative to the parameters is lost. Arnold finds a normal form M_A, depending on parameters, for an arbitrary family of complex matrices close to a given matrix M and depending holomorphically on parameters, such that both M_A and the reducing matrix g depend analytically on the same parameters. This is the simplest such normal form, in the sense that the number of parameters involved is as small as possible. The Arnold normal form M_A is computed as a versal deformation of the Jordan form M_J of the given matrix M. The transformation of M to the Arnold normal form M_A is a $GL(n, \mathbb{C})$-conjugacy transformation of the element M of the Lie algebra $gl(n, \mathbb{C})$ of the general linear group. Besides the case $M \in gl(n, \mathbb{C})$ studied by Arnold, versal deformations were also found for $M \in gl(n, \mathbb{R})$ (Galin 1972) and for the symplectic Lie algebras $sp(2n, \mathbb{C})$ and $sp(2n, \mathbb{R})$ (Galin 1975). In relativistic and quantum physics, the orthogonal groups play an important role. The Arnold normal forms of matrices M belonging to the Lie algebra $o(n, \mathbb{C})$ were also calculated (Patera and Rousseau 1982).

The purpose of this paper is to consider real semi-simple Lie algebras with involutions and to compute versal deformations of their elements. Representatives of conjugacy classes of elements of the algebras in a form analogous to the Jordan normal forms can be found either from Burgoyne and Cushman (1977) or explicitly from Djokovic *et al* (1981). These normal forms are again unstable. It is therefore natural to consider the problem of Arnold and Galin also for these algebras. More precisely, we consider the Lie algebras $o(p, q)$, $sp(2n, \mathbb{R})$, $u(p, q)$, $sp(2p, 2q)$, $o^*(2n)$, and find, in terms of versal deformation, a normal form of Arnold to which any family depending smoothly (analytically) on parameters can be reduced (in a neighbourhood of an element) under a smooth (analytic) conjugacy transformation from the corresponding Lie group.

The orthogonal and symplectic Lie algebras are treated simultaneously; they differ only by the value of a parameter ε ($\varepsilon = +1$ for $o(p, q)$ and $\varepsilon = -1$ for $sp(2n, \mathbb{R})$). For this reason and also because we do not use the same normal forms as Galin, we include here the case $sp(2n, \mathbb{R})$.

The set of parameter values which correspond to singularities for a family in general position is called the bifurcation diagram. Bifurcation diagrams for all $o(p, q)$ families of matrices depending on one or two parameters are computed.

The only real classical Lie algebra whose versal deformations have not been found is the linear algebra of quaternionic matrices $gl(n, \mathbb{H})$. After the work of Arnold and our treatment of quaternionic Lie algebras $o^*(2n)$ and $sp(2p, 2q)$, it is an easy straightforward matter.

In § 2 we recall the normal form of elements of real semi-simple Lie algebras with involutions. In § 3 the versal deformations are described. The results are summarised in the tables. The last section contains a description of bifurcation diagrams and many examples which are often encountered in publications. Namely, normal forms of elements of orthogonal Lie algebras $o(p, q)$, $p+q \leqslant 6$, are described and summarised in table 7; table 8 contains a list of all strata of $o(p, q)$, p and $q \geqslant 0$, with codimension 3 or less; a versal deformation of $o(p, q)$ elements of codimension 1 and 2 are given (table 9) together with corresponding bifurcation diagrams; finally, a versal deformation of the 'Lorentz boost generator' K_2 (cf equation (1.1)) is derived in all details.

2. Conjugacy classes of real Lie algebras

In this section we describe representatives of elements of conjugacy classes of the Lie algebras $o(p, q)$, $sp(2n, \mathbb{R})$, $u(p, q)$, $o^*(2n)$, $sp(2p, 2q)$. They are the analogous normal forms of the Jordan normal forms in these cases. The results are well known (Burgoyne and Cushman 1977, Djokovic *et al* 1981) and our presentation follows the second reference.

We denote by $\mathrm{L}(N, \mathbb{R})$ (where N could be $p+q$, $2n$ or $2p+2q$ depending on the case) any of the Lie algebras listed above and by $\mathrm{L}(N, \mathbb{R})$ the corresponding Lie group. $\mathrm{L}(N, \mathbb{R})$ is the group of k-linear operators on k^N (where k could be $\mathbb{R}$, $\mathbb{C}$ or the algebra $\mathbb{H}$ of quaternions depending on the case) preserving a given non-degenerate bilinear form f on k^N, i.e. if $g \in \mathrm{l}(N, \mathbb{R})$, then

$$\forall x, y \in k^N, \qquad f(gx, gy) = f(x, y). \tag{2.1}$$

The bilinear form f has a certain property of symmetry or of antisymmetry and it is represented by an invertible matrix K satisfying one of the relations

$$K^{\mathrm{T}} = \varepsilon K \qquad \text{where } \varepsilon = \begin{cases} +1 & \text{for } \mathrm{O}(p, q) \text{ and } \mathrm{Sp}(2p, 2q) \\ -1 & \text{for } \mathrm{Sp}(2n, \mathbb{R}) \text{ and } \mathrm{O}^*(2n) \end{cases} \tag{2.2}$$

$$K^+ = \varepsilon K \qquad \text{for } \mathrm{U}(p, q) \text{ with } \varepsilon = +1$$

(K^+ denotes the complex conjugate transpose of K). For the groups $\mathrm{O}(p, q)$, $\mathrm{U}(p, q)$ ($\mathrm{Sp}(2p, 2q)$) we have the additional property that the form is of signature (p, q) ($(2p, 2q)$). $\mathrm{L}(N, \mathbb{R})$ is represented as a subset of the set of non-singular matrices $g \in k^{N\times N}$, where

$$k = \begin{cases} \mathbb{R} & \text{for } \mathrm{O}(p, q) \text{ and } \mathrm{Sp}(2n, \mathbb{R}) \\ \mathbb{C} & \text{for } \mathrm{U}(p, q) \\ \mathbb{H} & \text{for } \mathrm{O}^*(2n) \text{ and } \mathrm{Sp}(2p, 2q). \end{cases} \tag{2.3}$$

The set $\mathbb{H}$ of quaternions is identified with the set of 2×2 matrices $\left(\begin{smallmatrix} \gamma & \beta \\ -\beta^* & \gamma^* \end{smallmatrix}\right)$ where γ, $\beta \in \mathbb{C}$, γ^* being the complex conjugate of γ. The quaternionic conjugate is defined as

$$\alpha\begin{pmatrix} \gamma & \beta \\ -\beta^* & \gamma^* \end{pmatrix} = \begin{pmatrix} \gamma^* & -\beta \\ \beta^* & \gamma \end{pmatrix} = \begin{pmatrix} \gamma & \beta \\ -\beta^* & \gamma^* \end{pmatrix}^+. \tag{2.4}$$

Identifying an element of $\mathrm{L}(N, \mathbb{R})$ with its matrix representation, we may write

$$g \in \mathrm{L}(N, \mathbb{R}) \qquad \text{iff } \forall x, y \in k^N, x\bar{g}Kg\bar{y} = xK\bar{y}, \tag{2.5}$$

where

$$\bar{a} = \begin{cases} a^{\mathrm{T}} \text{ if } a \text{ is a matrix over } \mathbb{R} \\ a^+ = (a_{ij}^*)^{\mathrm{T}} \text{ if } a = (a_{ij}) \text{ with } a_{ij} \in \mathbb{C} \\ (\alpha(a_{ij}))^{\mathrm{T}} \text{ if } a = (a_{ij}) \text{ with } a_{ij} \in \mathbb{H} \text{ and } \alpha \text{ given by (2.4).} \end{cases} \tag{2.6}$$

Equation (2.5) implies

$$g \in \mathrm{L}(N, \mathbb{R}) \qquad \text{iff } \bar{g}Kg = K. \tag{2.7}$$

A matrix $M \in k^{N\times N}$ belongs to the Lie algebra $\mathrm{l}(N, \mathbb{R})$ iff the 'infinitesimal equivalent' of (2.5) is satisfied i.e.

$$\forall x, y \in k^N, x\bar{M}K\bar{y} + xKM\bar{y} = 0 \tag{2.8}$$

which implies

$$M \in \mathrm{l}(N, \mathbb{R}) \qquad \text{iff } KM + \bar{M}K = 0. \tag{2.9}$$

Therefore, in order to specify an element of $\mathrm{L}(N, \mathbb{R})$, one has to give first a matrix K satisfying one of the conditions (2.2) with det $K \neq 0$, and a matrix M satisfying together with K the condition $KM + \bar{M}K = 0$. Our matrix K can change from case to case in order to obtain the simplest possible form for M (Burgoyne and Cushman 1977, Djokovic *et al* 1981). Two matrices K and K' satisfying the same one of the relations (2.2) and having the same signature are equivalent, i.e. $K' = gK\bar{g}$ for some $g \in \mathrm{GL}(N, k)$.

Two elements M and M' of $\mathrm{l}(N, \mathbb{R})$ belong to the same $\mathrm{L}(N, \mathbb{R})$-conjugacy class (or to the same orbit) iff

$$M' = gMg^{-1} \qquad \text{for some } g \in \mathrm{L}(N, \mathbb{R}). \tag{2.10}$$

The $\mathrm{L}(N, \mathbb{R})$ orbit of M is defined as

$$\mathrm{Orb}(M) = \{gMg^{-1} | g \in \mathrm{L}(N, \mathbb{R})\}. \tag{2.11}$$

Thus an orbit of M is a submanifold of $\mathrm{l}(N, \mathbb{R})$. In the case of the Jordan normal form of elements of $\mathrm{gl}(n, \mathbb{R})$, we have that any matrix is conjugate to a matrix which is a direct sum of irreducible blocks. Similarly here any matrix is conjugate to a direct sum of blocks of two types.

(i) Irreducible 'generalised Jordan blocks' (for the matrix elements we allow 2×2 matrices).

(ii) Blocks of the form $\begin{pmatrix} A & \\ & -\bar{A} \end{pmatrix}$ (with A a generalised Jordan block). Such a block is orthogonally indecomposable with respect to the form f.

A direct sum of blocks of the two types is said to be a generalised Jordan normal form.

Remark. Let us point out that the non-zero eigenvalues of any element of $\mathrm{l}(N, \mathbb{R})$ occur in pairs $\pm a$, except for the algebra $\mathrm{u}(p, q)$, where they occur in pairs α, $-\alpha^*$.

The indecomposable blocks are listed in table 1. In each case the corresponding matrix K is given. The following conventions are used in table 1:

$$\text{(i)} \qquad J_n = \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{pmatrix} \tag{2.12}$$

An arbitrary Jordan block is therefore denoted by $\alpha I_n + J_n$, where $\alpha \in \mathbb{C}$, and I_n is the identity $n \times n$ matrix.

(ii) We use tensorial notation for generalised Jordan blocks. For example, $\begin{pmatrix} a & b \\ -b & a \end{pmatrix} \otimes I_n + I_2 \otimes J_n$ represents the block

$$\begin{pmatrix} a & b & 1 & 0 & & & \\ -b & a & 0 & 1 & & & \\ & & \ddots & & \ddots & 1 & 0 \\ & & & \ddots & & 0 & 1 \\ & & & & \ddots & a & b \\ & & & & & -b & a \end{pmatrix}. \tag{2.13}$$

(iii) Cases come in pairs with the same matrix M_J and matrices K of opposite sign. They are distinguished by a parameter $\theta = \pm 1$, $\theta = 1$ for the unprimed cases and $\theta = -1$ for the others.

(iv) The matrix F_N is given by

$$F_N = \begin{pmatrix} & & & & 1 \\ & & & -1 & \\ & & 1 & & \\ & -1 & & & \\ \cdot^{\cdot^{\cdot}} & & & & \end{pmatrix}. \tag{2.14}$$

In particular,

$$F_2 = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}. \tag{2.15}$$

Table 1. Indecomposable Jordan normal forms of elements of the Lie algebras of $o(p, q)$, $sp(2n, \mathbf{R})$, $u(p, q)$, $sp(2p, 2q)$ and $o^*(2n)$. Matrices I_n, J_n and F_n are defined in § 2.

Algebra	Case	M_J	K	Signatures (p, q)
$o(p, q)$ $\varepsilon = +1$	I $(\theta = 1)$ I′ $(\theta = -1)$	$M_J = J_N$ $N = 2n$ if $\varepsilon = -1$ $N = 2n+1$ if $\varepsilon = +1$	$K = \theta F_N$	$p - q = \pm 1$
$sp(2n, \mathbf{R})$ $\varepsilon = -1$	II $(\theta = 1)$ II′ $(\theta = -1)$	$M_J = \begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix} \otimes I_n + \overset{\mathrm{i}}{I_2} \otimes J_n$ $b > 0$	$K = \theta \tilde{K} \otimes F_n$ $\tilde{K} = I_2 \begin{cases} n \text{ odd} & \varepsilon = +1 \\ n \text{ even} & \varepsilon = -1 \end{cases}$ $\tilde{K} = F_2 \begin{cases} n \text{ odd} & \varepsilon = -1 \\ n \text{ even} & \varepsilon = +1 \end{cases}$	→ $p - q = \pm 2$ → $p = q$
	III $(\theta = -1)$	$M_J = \begin{pmatrix} A & 0 \\ 0 & -A^T \end{pmatrix}$ $A = aI_n + J_n,\ a \geqslant 0$ n even if $\varepsilon = +1$ and $a = 0$ m odd if $\varepsilon = -1$ and $a = 0$	$K = \begin{pmatrix} 0 & I_n \\ \varepsilon I_n & 0 \end{pmatrix}$	$p = q$
	IV$(\theta = 1)$	$M_J = \begin{pmatrix} A & 0 \\ 0 & -A^T \end{pmatrix}$ $A = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \otimes I_n + I_2 \otimes J_n$ $a, b \in \mathbf{R},\ a > 0,\ b > 0$	$K = \begin{pmatrix} 0 & I_{2n} \\ \varepsilon I_{2n} & 0 \end{pmatrix}$	$p = q$
$u(p, q)$	I $(\theta = 1)$ I′ $(\theta = -1)$	$M_J = \mathrm{i}aI_n + J_n$	$K = \theta \mathrm{i} F_n$ if n even $K = \theta F_n$ if n odd	→ $p = q$ → $p - q = \pm 1$
$\varepsilon = +1$	II $(\theta = 1)$	$M_J = \begin{pmatrix} A & \\ & -A^+ \end{pmatrix}$ $A = \beta I_n + J_n \in \mathbf{C}$ $\operatorname{Re} \beta > 0$	$K = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}$	$p = q$

Table 1—(*continued*)

Algebra	Case	M_J	K	Signatures (p, q)
sp$(2p, 2q)$	I $(\theta = 1)$	$M_J = I_2 \otimes J_n$	$K = \theta\tilde{K} \otimes F_n$	
$\varepsilon = +1$	I′ $(\theta = -1)$		$\tilde{K} = I_2 \begin{cases} n \text{ odd} & \varepsilon = +1 \\ n \text{ even} & \varepsilon = -1 \end{cases}$	$\rightarrow$ $p - q = \pm 2$ (n odd)
			$\tilde{K} = F_2 \begin{cases} n \text{ odd} & \varepsilon = -1 \\ n \text{ even} & \varepsilon = +1 \end{cases}$	$\rightarrow$ $p = q$ (n even)
o*$(2n)$				
$\varepsilon = -1$	II $(\theta = 1)$	$M_J = \begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix} \otimes I_n + I_2 \otimes J_n$	$K = \theta\tilde{K} \otimes F_n$	
	II′ $(\theta = -1)$		$\tilde{K} = I_2 \begin{cases} n \text{ odd} & \varepsilon = +1 \\ n \text{ even} & \varepsilon = -1 \end{cases}$	$\rightarrow$ $p - q = \pm 2$ (n odd)
		$b > 0$	$\tilde{K} = F_2 \begin{cases} n \text{ odd} & \varepsilon = -1 \\ n \text{ even} & \varepsilon = +1 \end{cases}$	$\rightarrow$ $p = q$ (n even)
	III $(\theta = 1)$	$M_J = \begin{pmatrix} A & 0 \\ 0 & -A^{\mathrm{T}} \end{pmatrix}$	$K = \begin{pmatrix} & I_{2n} \\ \varepsilon I_{2n} & \end{pmatrix}$	$p = q$
	III′ $(\theta = -1)$	$A = aI_{2n} + I_2 \otimes J_n$		
		$a > 0$		
	IV $(\theta = 1)$	$M_J = \begin{pmatrix} A & 0 \\ 0 & -A^{\mathrm{T}} \end{pmatrix}$	$K = \begin{pmatrix} 0 & I_{2n} \\ \varepsilon I_{2n} & 0 \end{pmatrix}$	$p = q$
		$A = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \otimes I_n + I_2 \otimes J_n$		
		$a > 0, b > 0$		

Therefore $K = \tilde{K} \otimes F_N$ is the matrix

$$\begin{pmatrix} & & & & \tilde{K} \\ & & & -\tilde{K} & \\ & & \tilde{K} & & \\ & -\tilde{K} & & & \\ \cdot^{\cdot^{\cdot}} & & & & \end{pmatrix} \tag{2.16}$$

The Jordan normal form of a decomposable element of $\mathrm{l}(N, \mathbb{R})$ is a direct sum of normal forms of indecomposable elements of $\mathrm{l}(N, \mathbb{R})$, $(M_J^{(i)}, K^{(i)})$

$$M_J = \bigotimes_{i=1}^{s} M_J^{(i)} \qquad K = \bigotimes_{i=1}^{s} K^{(i)}. \tag{2.17}$$

3. Versal deformations of elements of $\mathrm{l}(N, \mathbb{R})$

Here we shall define the notion of versal deformation of an element of a real Lie algebra. More details about such deformations are to be found in (Arnold 1971).

Definition. (1) A *deformation* of an element $M_0 \in \mathrm{l}(N, \mathbb{R})$ is a family $M: \Lambda \to \mathrm{l}(N, \mathbb{R})$ where Λ is a neighbourhood of 0, $\Lambda \subseteq R^k$, depending smoothly (analytically) on the parameters and such that $M(0) = M_0$. (2) A deformation $M(\lambda)$ of M_0 is *versal* iff for any deformation $M'(\mu_1, \ldots, \mu_m) \in \mathrm{l}(N, \mathbb{R})$ of M_0, $M'(\mu)$ is induced by $M(\lambda)$, i.e. there exists a neighbourhood V of 0 in $\mathbb{R}^m$, a map $\varphi: V \to \mathbb{R}^k$ with $\varphi(0) = 0$, and a map $g: V \to \mathrm{L}(N, \mathbb{R})$ with $g(0) = I$ such that $\forall \mu \in V$, $M'(\mu) = g(\mu) M(\varphi(\mu)) g^{-1}(\mu)$, and both φ and g are smooth (analytic).

Obviously, it is enough to consider versal deformations of matrices M_J which are in generalised Jordan form and we consider only such matrices.

A technique for finding versal deformations is given by the following lemma.

Lemma. (Arnold 1971). A deformation $M(\lambda)$ of a matrix M_0 is versal iff $M(\lambda)$ is transversal to $\mathrm{Orb}\, M_0$ at the point $\lambda = 0$. The minimal number of parameters $\lambda = (\lambda_1, \ldots, \lambda_k)$ appearing in such a deformation is therefore equal to the codimension of $\mathrm{Orb}\, M_0$.

Definition. A versal deformation with the minimal number of parameters is called a *minimal versal deformation.*

By the Lemma the problem of computing versal deformations is reduced to the computation of a complement of the tangent space of $\mathrm{Orb}\, M_J$ at M_J in $\mathrm{l}(N, \mathbb{R})$. Among the complements of $T_{M_J} \mathrm{Orb}\, M_J$ we specify two: (1) the orthogonal complement under the inner product $(A, B) = \mathrm{Tr}(AB^+)$, (2) a complement which has the minimal number of non-zero entries. For the first one, we use the following lemma.

Lemma. (Arnold 1971) $C \in \mathrm{l}(N, \mathbb{R})$ is orthogonal to $\mathrm{Orb}(M_0)$ iff $[C^+, M_0] = 0$.

Therefore, one has to compute centralisers of elements of $\mathrm{l}(N, \mathbb{R})$.

Remark. The centralisers of decomposable elements are direct sums of centralisers of each block whenever there are no common eigenvalues of the blocks. In the other cases the centralisers have more complex structures but one can easily convince oneself that it is enough to compute centralisers of direct sums of pairs of indecomposable elements.

The centralisers in $\mathrm{l}(N, \mathbb{R})$ are the sets

$$\mathrm{Centr}_{\mathrm{l}(N,\mathbb{R})} M_J = \{C \| [C, M_J] = 0 \text{ and } KC + \bar{C}K = 0\} \tag{3.1}$$

where $\bar{C}$ is given in (2.6). The centralisers are given by figures 1–5 in table 2 together with the explanations appearing in tables 3, 4 or 5 depending on the algebra.

Theorem 1. Let $M_J \in \mathrm{l}(N, \mathbb{R})$ be in generalised Jordan normal form. Then $M_\Lambda^{(1)} = M_J + C^+(\lambda)$ is a minimal versal deformation of M_J, where $C(\lambda)$ is a generic element of $\mathrm{Centr}_{\mathrm{l}(N,\mathbb{R})} M_J$. The number d of independent parameters is equal to the dimension of $\mathrm{Centr}_{\mathrm{l}(N,\mathbb{R})} M_J$ and is computed according to the last column of tables 3–5.

Proof. It follows from (2.9) and the special form of the matrix K that if $C \in \mathrm{l}(N, \mathbb{R})$ then $C^+ \in \mathrm{l}(N, \mathbb{R})$.

An explicit global formula for d can be found in Patera *et al* (1982).

The versal deformation of theorem 1 is orthogonal to the orbit of M_J; however, it contains many non-zero matrix elements. In a normal form one would like to have as

Table 2. Centralisers of normal forms of elements of $\mathrm{l}(N, \mathbb{R})$. All full lines denote a repetition of the same element along the line, the element being one of the following types: $c \in \mathbb{R}$, $id \in i\mathbb{R}$, $\gamma \in \mathbb{C}$, $\left(\begin{smallmatrix} c & d \\ -d & c \end{smallmatrix}\right)$ with $c, d \in \mathbb{R}$, $\left(\begin{smallmatrix} \gamma & \beta \\ -\beta^* & \gamma^* \end{smallmatrix}\right) \in \mathbb{H}$ etc. For any matrix $C = (c_{ij})$, the matrix $\bar{C}$ is defined in (2.6). ε is given for the different algebras in tables 3–5. All broken lines represent a repetition of an element of the types indicated but with signs alternating from place to place.

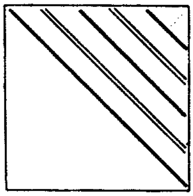


Figure 1

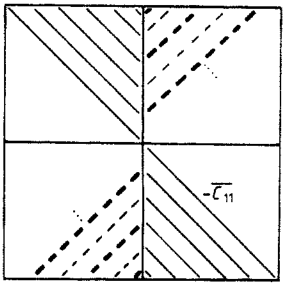


Figure 2

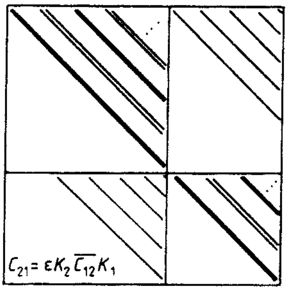


Figure 3

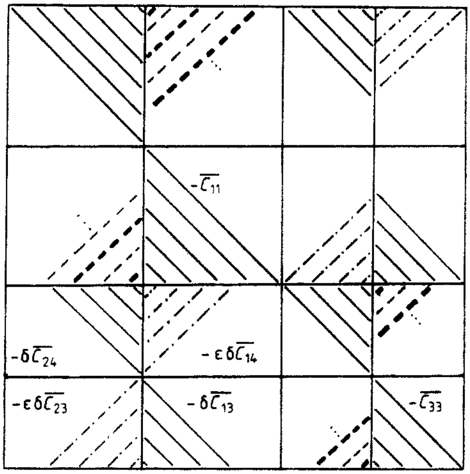


Figure 4

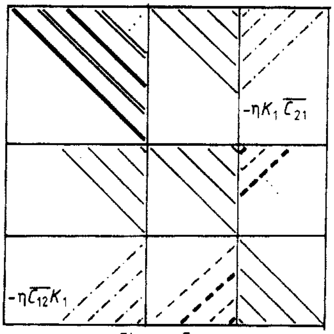


Figure 5

many zero matrix elements as possible. For that we compute the general form of a tangent vector to $\mathrm{Orb}(M_J)$, i.e. a vector D of the form $[D, M_J]$ with $D \in \mathrm{l}(N, \mathbb{R})$. The general form of an element $[D, M_J]$ is related to the general form of the adjoint C^+ of an element $C \in \mathrm{Centr}_{\mathrm{l}(N,\mathbb{R})} M_J$. Corresponding to each full (broken) line of C^+ which is filled with non-zero elements in tables 2–5 we have that the sum (alternating sum) of the

Table 3. Minimal versal deformations of elements of $o(p, q)$ and $sp(2n, \mathbb{R})$. Details of notation in table 2. Roman (Greek) letters denote real (complex) numbers. The centraliser column gives the legend for the figures of table 2. For decomposable cases, for example I + III, the legend of the diagonal blocks is found next to the simpler cases I, (III). In I + III one only finds the legend for the non-diagonal blocks. In the column 'order of M_J', for the decomposable cases, we give the respective orders of the two blocks mentioned in the column 'case'. The last column gives the minimal number of real parameters for a versal deformation. A matrix C in the centraliser is thought of as a matrix of blocks i.e. $C = (C_{ij})$ where C_{ij} is a matrix.

Case eigenvalues	Centraliser	$M(\lambda)$	Order of M_J	d
I 0	Figure 1 —— line of 0 ═══ $c \in \mathbb{R}$	$\varepsilon = +1$ Figure 1″ $\lambda_i = \nu_i = 0$ $\mu_i = \xi_i \in \mathbb{R}$ $\varepsilon = -1$ Figure 1′ $\lambda_i \in \mathbb{R}$ $\mu_i = \nu_i = 0$	$N = 2n + 1$ or $N = 2n$	n
II $(+ib)$	Figure 1 —— $\begin{pmatrix} 0 & c \\ -c & 0 \end{pmatrix}$ ═══ $\begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix}$	$\tilde{K} = F_2$ n odd Figure 1′ $\lambda_i \sim \begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix}$ $\mu_i = \begin{pmatrix} 0 & 0 \\ 0 & d_i \end{pmatrix}$ $\nu_i = \begin{pmatrix} d_i & 0 \\ 0 & 0 \end{pmatrix}$ n even Figure 1″ $\lambda_i = \begin{pmatrix} 0 & 0 \\ c_i & 0 \end{pmatrix} = +\nu_i$ $\mu_i = \begin{pmatrix} 0 & 0 \\ 0 & d_i \end{pmatrix}$ $\xi_i = \begin{pmatrix} d_i & 0 \\ 0 & 0 \end{pmatrix}$ $\tilde{K} = I_2$ Figure 1″ $\lambda_i \sim \begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix}$ $\nu_i = -\lambda_i^{T}$ $\xi_i = \mu_i \sim \begin{pmatrix} 0 & 0 \\ 0 & d \end{pmatrix}$ In all cases the left bottom corner is the sum of what should happen there	$2n$	n

Table 3—*continued*

Case eigenvalues	Centraliser	$M(\lambda)$	Order of M_J	d
III	Figure 2 —— $c \in \mathbb{R}$	Figure 2′ $\lambda_i \in \mathbb{R}$	$2n$	$a \neq 0$ n
$(\pm a)$	$a \neq 0$ $C_{12} = C_{21} = 0$	$a \neq 0$ $\mu_i = \zeta_i = \nu_i = \xi_i = 0$		$a = 0 \begin{cases} 2n - \varepsilon & n \text{ odd} \\ 2n & n \text{ even} \end{cases}$
	$a = 0$ $\varepsilon = +1$ --- line of 0 ---- $c \in \mathbb{R}$	$a = 0$ $\varepsilon = +1$ $\nu_i = \xi_i = 0$ $\mu_i, \zeta_i \in \mathbb{R}$		
	$\varepsilon = -1$ interchange --- and ---	$\varepsilon = -1$ $\nu_i, \xi_i \in \mathbb{R}$ $\mu_i = \zeta_i = 0$		
IV	Figure 2 —— $\begin{pmatrix} c & d \\ -d & c \end{pmatrix}$ --- lines of 0	Figure 2′ $\lambda_i \in \mathbb{R}$		
$(\pm a \pm ib)$		$\mu_i = \zeta_i = \nu_i = \xi_i = 0$	$4n$	$2n$
$\left.\begin{matrix} \text{I} + \text{I} \\ \text{I}' + \text{I}' \end{matrix}\right\} \delta = 1$	Figure 3 —— $c \in \mathbb{R}$	Figure 3′ $\mu_i = \nu_i \in \mathbb{R}$	$N_1 \geqslant N_2$	$n_1 + n_2 + N_2$
I + I′ $\delta = -1$				
I + III $\delta = 1$	Figure 5 —·— $c \in \mathbb{R}$	Figure 5′ $\sigma_i, \tau_i \in \mathbb{R}$	$N_1, 2n_2$	$n_1 + 2n_2 + 2\min(N_1, n_2) - \varepsilon$ for n_2 odd
I′ + III $\delta = -1$ (with $a = 0$)				$n_1 + 2n_2 + 2\min(N_1, n_2)$ for n_2 even

III + III	Figure 4 $a=0$ $-\cdot-$ $c\in\mathbb{R}$ $a\neq 0$ $C_{ij}=0$ for $i+j$ odd	Figure 4′ $a=0$ $\varphi_i, \chi_i, \psi_i, \omega_i \in\mathbb{R}$ $a\neq 0$ $\varphi_i, \omega_i\in\mathbb{R}$ $\chi_i=\psi_i=0$	$2n_1\geqslant 2n_2$	$a\neq 0$ n_1+3n_2 $a=0$ $2n_1+6n_2-\varepsilon$ iff n_1+n_2 odd $2n_1+6n_2-2\varepsilon$ iff n_1n_2 odd $2n_1+6n_2$ if n_1, n_2 even
II + II, II′ + II′ $\delta=1$ II + II′ $\delta=-1$	Figure 3 $-$ $\begin{pmatrix} c & d \\ -d & c \end{pmatrix}$	Figure 3′ $\tilde{K}_1=\tilde{K}_2=F_2$ $\mu_i=-\nu_i\in\mathbb{R}$ $\tilde{K}_1=\tilde{K}_2=I_2$ $\mu_i\sim\begin{pmatrix} 0 & 0 \\ c & d \end{pmatrix}$ $\nu_i=\mu_i^{\mathrm{T}}$ $\tilde{K}_1=I_2$ $\tilde{K}_2=F_2$ $\mu_i=\begin{pmatrix} 0 & 0 \\ c_i & d_i \end{pmatrix}$ $\nu_i=\begin{pmatrix} -c_i & 0 \\ -d_i & 0 \end{pmatrix}$ $\tilde{K}_1=F_2$ $\tilde{K}_2=I_2$ $\mu_i=\begin{pmatrix} 0 & 0 \\ c_i & d_i \end{pmatrix}$ $\nu_i=\begin{pmatrix} 0 & d_i \\ 0 & -c_i \end{pmatrix}$	$2n_1\geqslant 2n_2$	n_1+3n_2
IV + IV	Figure 4 $C_{ij}=0$ for $i+j$ odd	Figure 4′ $\varphi_i, \omega_i\in\mathbb{R}$ $\chi_i=\psi_i=0$	$4n_1\geqslant 4n_2$	$2n_1+6n_2$

Table 4. Minimal versal deformations of elements of $u(p, q)$. Details of notation in tables 2 and 3.

Case eigenvalues	Centraliser	$M(\lambda)$	Order of M_J	d
I ib	Figure 1 —— $ic \in i\mathbb{R}$ ===== $d \in \mathbb{R}$	Figure 1′ N odd $\lambda_j \in i\mathbb{R}$ $\mu_j = \nu_j \in \mathbb{R}$ N even $\lambda_j \in \mathbb{R}$ $\mu_j = \nu_j \in i\mathbb{R}$	N	N
II $\beta, -\beta^*$	Figure 2 —— $\gamma \in \mathbb{C}$ $C_{12} = C_{21} = 0$	Figure 2′ $\lambda_i \in \mathbb{C}$ $\mu_i = \nu_i = \xi_i = \zeta_i = 0$	$2n$	$2n$
$\left.\begin{matrix}\mathrm{I+I}\\ \mathrm{I'+I'}\end{matrix}\right\}\delta = 1$ I+I′ $\delta = -1$	Figure 3 —— $\gamma \in \mathbb{C}$	Figure 3′ $\mu_i \in \mathbb{C}$ $\nu_i = \mu_i^*$ N_1, N_2 even $\delta = -1$ N_1, N_2 odd $\delta = +1$ $N_1 + N_2$ odd $\delta = +i$ The effect of this δ is added to the effect of the δ in the column 'case'	$N_1 \geqslant N_2$	$N_1 + 3N_2$
$\left.\begin{matrix}\mathrm{II+II}\\ \mathrm{II'+II'}\end{matrix}\right\}\delta = 1$ II+II′ $\delta = -1$	Figure 4 $C_{ij} = 0$ for $i+j$ odd	Figure 4′ $\varphi_i, \omega_i \in \mathbb{C}$ $\chi_i = \psi_i = 0$	$2n_1 \geqslant 2n_2$	$2n_1 + 6n_2$

Table 5. Minimal versal deformations of elements of sp(2p, 2q) and o*(2n). Details of notation in table 2.

Case eigenvalues	Centraliser	$M(\lambda)$	Order of M_J	d
I 0	Figure 1 $\tilde{K}=I_2$ —— $\begin{pmatrix} \mathrm{i}c & \beta \\ -\beta^* & -\mathrm{i}c \end{pmatrix}$	Figure 1′ $\tilde{K}=I_2$	$2n$	$2n$
	$=\!=\begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix}$	n odd $\lambda_i \sim \begin{pmatrix} \mathrm{i}c & \beta \\ -\beta^* & -\mathrm{i}c \end{pmatrix}$ $\mu_i=\nu_i \sim \begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix}$		
	$\tilde{K}=F_2$ —— $\begin{pmatrix} 0 & d \\ -d & 0 \end{pmatrix}$	n even $\lambda_i \sim \begin{pmatrix} 0 & d \\ -d & 0 \end{pmatrix}$ $\mu_i=\nu_i \sim \begin{pmatrix} \gamma & \mathrm{i}d \\ \mathrm{i}d & \gamma^* \end{pmatrix}$		
	$\begin{pmatrix} \gamma & \mathrm{i}d \\ \mathrm{i}d & \gamma^* \end{pmatrix}$	$\tilde{K}=F_2$ n odd $\lambda_i \sim \begin{pmatrix} 0 & d \\ -d & 0 \end{pmatrix}$ $\mu_i=\nu_i \sim \begin{pmatrix} \gamma & \mathrm{i}c \\ \mathrm{i}c & \gamma^* \end{pmatrix}$		
		n even $\lambda_i \sim \begin{pmatrix} \gamma & \mathrm{i}c \\ \mathrm{i}c & \gamma^* \end{pmatrix}$ $\mu_i=\nu_i \sim \begin{pmatrix} 0 & d \\ -d & 0 \end{pmatrix}$		
II	Figure 1 —— $\begin{pmatrix} 0 & d \\ -d & 0 \end{pmatrix}$	Figure 1′ n odd $\lambda_i \sim \begin{pmatrix} 0 & d \\ -d & 0 \end{pmatrix}$ $\mu_i=\nu_i \sim \begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix}$	$2n$	n
$+\mathrm{i}b$	$=\!=\begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix}$	n even $\lambda_i \sim \begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix}$ $\mu_i=\nu_i \sim \begin{pmatrix} 0 & d \\ -d & 0 \end{pmatrix}$		
III	Figure 2 —— $\begin{pmatrix} \gamma & \beta \\ -\beta^* & \gamma^* \end{pmatrix}$	Figure 2′ $\lambda_i \sim \begin{pmatrix} \gamma & \beta \\ -\beta^* & \gamma^* \end{pmatrix}$	$4n$	$4n$
$(\pm a)$	$=\!=\!=$ lines of 0	$\mu_i=\nu_i=\zeta_i=\xi_i=0$		

Table 5—*continued*

Case eigenvalues	Centraliser	$M(\lambda)$	Order of M_I	d
IV	Figure 2 —— $\begin{pmatrix} c & d \\ -d & c \end{pmatrix}$	Figure 2′ $\lambda_i \sim \begin{pmatrix} c & d \\ -d & c \end{pmatrix}$	$4n$	$2n$
$(\pm a \pm ib)$	=== lines of 0	$\mu_i = \nu_i = \zeta_i = \xi_i = 0$		
$\left.\begin{matrix} \mathrm{I+I} \\ \mathrm{I'+I'} \end{matrix}\right\} \delta = 1$ $\mathrm{I+I'}$ $\delta = -1$	Figure 3 —— $\begin{pmatrix} \gamma & \beta \\ -\beta^* & \gamma^* \end{pmatrix}$	Figure 3′ $\mu_i \sim \begin{pmatrix} \gamma & \beta \\ -\beta^* & \gamma^* \end{pmatrix}$ $\tilde{K}_1 = \tilde{K}_2 = I_2 \quad \nu_i = \alpha(\mu_i)$ $\tilde{K}_1 = I_2 \quad \tilde{K}_2 = F_2 \quad \nu_i = \alpha(\mu_i)\tilde{K}_2$ $\tilde{K}_1 = F_2 \quad \tilde{K}_2 = I_2 \quad \nu_i = \tilde{K}_1\alpha(\mu_i)$ $\tilde{K}_1 = \tilde{K}_2 = F_2 \quad \nu_i = -\mu_i^{\mathrm{T}}$	$2n_1 \geq 2n_2$	$2n_1 + 6n_2$
$\left.\begin{matrix} \mathrm{II+II} \\ \mathrm{II'+II'} \end{matrix}\right\} \delta = 1$ $\mathrm{II+II'}$ $\delta = -1$	Figure 3 —— $\begin{pmatrix} c & d \\ -d & c \end{pmatrix}$	Figure 3′ $\mu_i \sim \begin{pmatrix} c & d \\ -d & c \end{pmatrix}$ ν_i is related to μ_i as in I + I	$2n_1 \geq 2n_2$	$n_1 + 3n_2$
III + III	Figure 4 - - - $\begin{pmatrix} \gamma & \beta \\ -\beta^* & \gamma^* \end{pmatrix}$ $C_{ij} = 0$ for $i + j$ odd	Figure 4′ $\chi_i = \psi_i = 0 \quad \varphi_i, \omega_i \sim \begin{pmatrix} \gamma & \beta \\ -\beta^* & \gamma^* \end{pmatrix}$	$4n_1 \geq 4n_2$	$4n_1 + 12n_2$
IV + IV	Figure 4 - - - $\begin{pmatrix} c & d \\ -d & c \end{pmatrix}$ $C_{ij} = 0$ for $i + j$ odd	Figure 4′ $\chi_i = \psi_i = 0 \quad \varphi_i, \omega_i \sim \begin{pmatrix} c & d \\ -d & c \end{pmatrix}$	$4n_1 \geq 4n_2$	$2n_1 + 6n_2$

elements on the corresponding line of $[D, M_J]$ is zero, and when the line is made up of blocks $\left(\begin{smallmatrix} a & b \\ -b & a \end{smallmatrix}\right)$ the sum (alternating sum) of the corresponding blocks in $[D, M_J]$ is of the form $\left(\begin{smallmatrix} c & d \\ d & -c \end{smallmatrix}\right)$.

In order to find a complement to the tangent space to Orb M_J, one keeps in mind the general form of C^+ where $C \in$ Centr M_J and takes the elements A of $\mathrm{l}(N, \mathbb{R})$ which do not have the property that 0 equals the sum (alternating sum) of a line of A corresponding to a full (broken) line of C^+ in case this line is filled with non-zero elements.

If a line of C^+ is made up of blocks $\left(\begin{smallmatrix} a & b \\ -b & a \end{smallmatrix}\right)$ then one takes A such that the sum (alternating sum) of the corresponding blocks of A is not of the form $\left(\begin{smallmatrix} c & d \\ d & -c \end{smallmatrix}\right)$. For example, in order to break the property that the sum (alternating sum) of the elements on the line is zero, we repeat the same element (the element with alternating sign) on each place on the line. Other solutions are found by taking on the line only the minimal number of elements necessary to break the property. For $\mathrm{gl}(n, \mathbb{C})$ (Arnold 1971) it suffices to take one element on each line. Here we sometimes have to take several non-zero elements in order to stay inside the algebra $\mathrm{l}(N, \mathbb{R})$.

Theorem 2. Let $M_J \in \mathrm{l}(N, \mathbb{R})$ be in generalised Jordan normal form. Then $M_A^{(2)} = M_J + M(\lambda)$ is a minimal versal deformation of M_J, where $M(\lambda)$ is a generic element of a transversal subspace of T_{M_J} Orb M_J in $\mathrm{l}(N, \mathbb{R})$, given by the figures of table 6 and the legend of tables 2–5.

Theorem 3. Let $M(\mu_1, \ldots, \mu_m)$ be a C^∞ family of elements of $\mathrm{l}(N, \mathbb{R})$ with $M(0) = M_0$ and let M_J be the generalised Jordan normal form of M_0. Let $d = \operatorname{codim}$ Orb M_J. Then there exist C^∞ mappings

$$\varphi: \mathbb{R}^m \to \mathbb{R}^d \qquad g: \mathbb{R}^m \to \mathrm{L}(N, \mathbb{R})$$

such that $\varphi(0) = 0$ and $M(\mu) = g(\mu) M_A(\varphi(\mu)) g^{-1}(\mu)$, where M_A is a versal deformation of M_J, given in either of theorems 1, 2.

4. Bifurcation diagrams and examples

In this section we consider families of elements of $\mathrm{l}(N, \mathbb{R})$ in general position (the definition is given below) and investigate the possible structures of the matrices of a family. Then a number of examples involving the Lie algebras $\mathrm{o}(p, q)$ are considered.

Definition. Two elements of $\mathrm{l}(N, \mathbb{R})$ have the same *structure* or, equivalently, belong to the same *stratum*, iff their centralisers are conjugate under the action of $\mathrm{L}(N, \mathbb{R})$.

Considering the centralisers (tables 2–5) of the normal forms of elements of the different algebras $\mathrm{l}(N, \mathbb{R})$, it is straightforward to prove the following propositions.

Proposition 1 ($\mathrm{sp}(2n, \mathbb{R})$). Two elements of $\mathrm{sp}(2n, \mathbb{R})$ belong to the same stratum iff their Jordan normal forms (M_J, K) have:

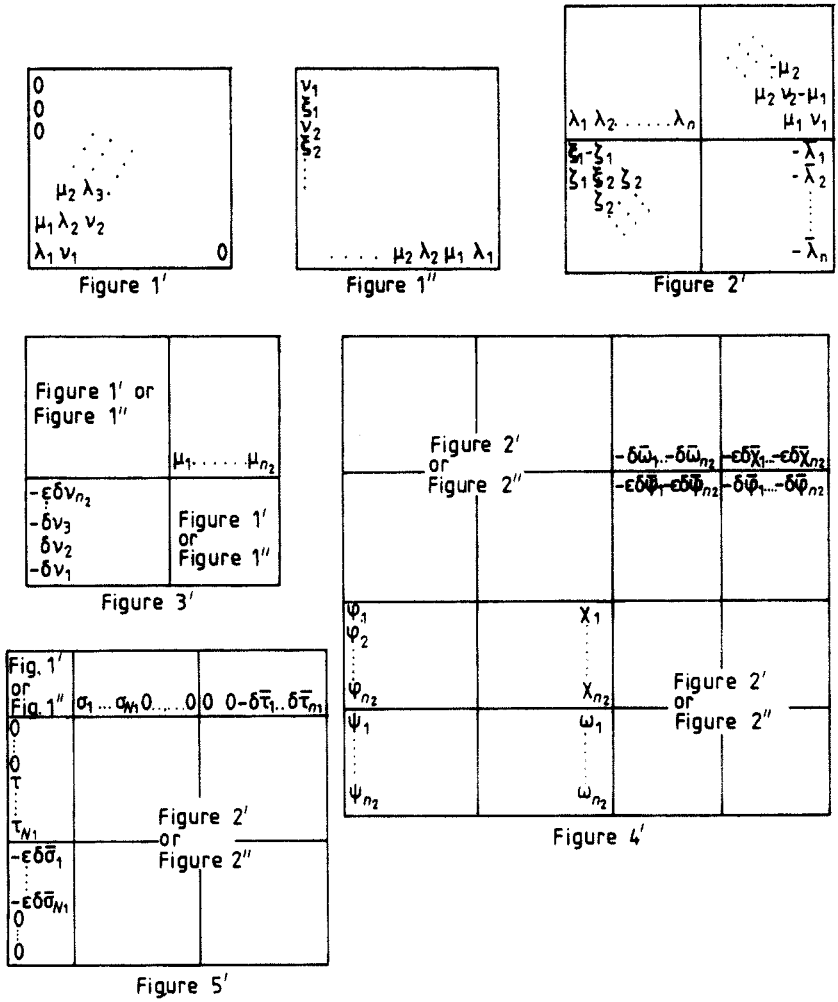
(i) the same number of pairs of non-zero real eigenvalues $(\pm a)$ and for each pair the same blocks of type III,

(ii) the same number of quadruples of non-zero eigenvalues $(\pm a \pm ib)$ and for each quadruple the same blocks of type IV,

(iii) the same number of pairs of non-zero eigenvalues $(\pm ib)$, $b \in \mathbb{R}$ and for each pair, the same blocks of type II,

(iv) the same blocks of types I and III for the zero eigenvalue.

Table 6. Transversal subspaces to orbits of normal forms of elements of $\mathrm{l}(N,\mathbb{R})$ with minimal number of non-zero entries. The quantities λ_j, μ_j, etc of table 4 are of the form $c\in\mathbb{R}$, $ic\in i\mathbb{R}$, $\left(\begin{smallmatrix} c & d\\ -d & c\end{smallmatrix}\right)$, $\left(\begin{smallmatrix} \gamma & \beta\\ -\beta^* & \gamma^*\end{smallmatrix}\right)\in\mathbb{H}$ etc. $\lambda_i\sim\left(\begin{smallmatrix} c & d\\ -d & c\end{smallmatrix}\right)$ means: λ_i is of the form $\left(\begin{smallmatrix} c & d\\ -d & c\end{smallmatrix}\right)$. $\bar\sigma$ is given as in (2.6). F_2 is given in (2.15).



Proposition 2 (sp(2*p*, 2*q*)). Two elements of sp(2*p*, 2*q*) belong to the same stratum iff their Jordan normal forms (M_J, K) have:

(i) the same number of pairs of non-zero real eigenvalues $(\pm a)$ and for each pair the same blocks of type III,

(ii) the same number of quadruples of non-zero eigenvalues $(\pm a\ \pm ib)$ and for each quadruple the same blocks of type IV,

(iii) the same number of pairs of non-zero eigenvalues $(\pm ib)$, $b\in\mathbb{R}$, and for each pair, the same blocks of type II, with the corresponding part of the matrix K of same signature,

(iv) the same blocks of types I and III for the zero eigenvalue with the corresponding part of the matrix K of same signature.

Proposition 3 (u(p, q)). Two elements of u(p, q) have the same structure iff their normal forms (M_J, K) have:

(i) the same number of eigenvalues ib and for each eigenvalue ib the same blocks of type I, with the corresponding part of the matrix K of same signature,

(ii) the same number of pairs of eigenvalues (β, $-\beta^*$) and for each pair the same blocks of type II.

Proposition 4 (o(p, q)). Two matrices of o(p, q) belong to the same stratum iff their normal forms (M_J, K) have:

(i) the same number of pairs of eigenvalues $\pm a_1, \ldots, \pm a_s, \pm ib_1, \ldots, \pm ib_r$ with the a_j (b_j) distinct and either $\pm a_j$ ($\pm ib_j$) are simple non-zero eigenvalues or $a_j = 0$ ($b_j = 0$) and the eigenvalue 0 has multiplicity 2,

(ii) the same number of multiple non-zero eigenvalues ($\pm a$) and for each of these pairs the same blocks of type III,

(iii) the same number of multiple non-zero eigenvalues ($\pm ib$) and for each of these pairs, the same blocks of type II with the corresponding part of the matrix K of same signature,

(iv) the same number of quadruple eigenvalues ($\pm a$ $\pm ib$) and for each of these quadruples the same blocks of type IV,

(v) the same blocks of types I and III for the eigenvalue 0, when 0 has multiplicity not equal to 2, with the corresponding part of K of the same signature.

Proposition 5 (o*($2n$)). Two matrices of o*($2n$) belong to the same stratum iff their normal forms (M_J, K) have:

(i) the same number of pairs of eigenvalues $\pm ib_1, \ldots, \pm ib_s$, with the b_i distinct and either ib_j are simple non-zero eigenvalues or $b_j = 0$ and the eigenvalue 0 has multiplicity 2,

(ii) the same number of eigenvalues ($\pm a$) and for each of these pairs the same blocks of type III,

(iii) the same number of multiple eigenvalues ($\pm ib$) and for each of these pairs the same blocks of type II,

(iv) the same number of eigenvalues ($\pm a$ $\pm ib$) and for each of these quadruples the same blocks of type IV,

(v) the same blocks of type I for the eigenvalue 0, when 0 has multiplicity not equal to 2.

Each stratum is a semi-algebraic manifold (defined by equalities and inequalities).

The splitting of l(N, $\mathbb{R}$) into strata is a finite semi-algebraic stratification. Consequently, the transversality theorem (Thom and Levine 1959) applies to it. Therefore one has the following.

Corollary. The set of families of elements of l(N, $\mathbb{R}$) transversal to all the strata is a dense set (which is a countable intersection of dense open sets).

From this the following definition becomes very natural.

Definition. A family of matrices is in *general position* if it is transversal to all the strata. The bifurcation diagram of such a family is the set of parameter values which correspond to matrices belonging to strata of positive codimension. A stratum of positive codimension is called a *singularity*.

The bifurcation diagram is a finite union of smooth manifolds, each of which corresponds to a set of matrices with a given structure. The codimension of a stratum in $\mathrm{l}(N, \mathbb{R}) \neq \mathrm{u}(p, q)$ is (Patera *et al* 1982):

$$c = d - \tfrac{1}{2}\nu - \rho \qquad \begin{array}{ll} \rho = +1 & \text{when 0 has multiplicity 2 and } \mathrm{l}(N, \mathbb{R}) = \mathrm{o}(p, q) \text{ or } \mathrm{o}^*(2n) \\ \rho = 0 & \text{otherwise} \end{array} \tag{4.1}$$

where ν is the number of distinct non-zero eigenvalues of the stratum. For a stratum in $\mathrm{u}(p, q)$ the codimension is given by

$$c = d - \nu \tag{4.2}$$

where ν is the number of distinct eigenvalues. In (4.1) and (4.2), d is the codimension of the orbit of any element of the stratum. It is given in tables 3–5.

Notation. Irreducible blocks of types I, II, III, IV of elements of $\mathrm{l}(N, \mathbb{R}) \neq \mathrm{u}(p, q)$ are denoted by 0^n, $(\pm ib)^n$, $(\pm a)^n$ ($(\pm 0)^n$ if $a = 0$), $(\pm a \pm ib)^n$. When it is necessary to mention the signature of the corresponding part of K we use the subscripts $\pm$:

0^n_+, $(\pm ib)^n_+$ if $p > q$ (depending on θ in table 1)

0^n_-, $(\pm ib)^n_-$ if $p < q$.

In the case of $\mathrm{u}(p, q)$, blocks of types I and II are denoted respectively $(ib)^n$ and $(\beta, -\beta^*)^n$. When necessary, we distinguish between $(ib)^n_+$ for $p > q$ and $(ib)^n_-$ for $p < q$.

Here n is the multiplicity of the eigenvalue. A stratum is denoted by the product of its blocks. The signature of each block is the signature of the corresponding K in table 1. For many examples see tables 7 and 8.

Table 7 contains all normal forms M_J of $\mathrm{o}(p, q)$, $p + q \leq 6$. Each entry represents a stratum. An entry with non-zero eigenvalues represents a continuum of orbits of the stratum distinguished by their eigenvalues. Two entries belonging to the same $\mathrm{o}(p, q)$ are given relative to *different* bases and can be compared only after transformation to a common basis (cf an example below). Together with each entry the table contains also the codimensions d and c of its orbit and stratum.

Instead of listing all the normal forms of elements of a given Lie algebra, it is more economical to list elements of the same codimensions for Lie algebras of the same type and all ranks because simple eigenvalues do not contribute to c. Often the low codimensions are the most interesting ones. Table 8 contains a list of all strata in $\mathrm{o}(p, q)$ with codimension $c \leq 3$. Each stratum is identified by a representative of its elements. The underlined entries of codimensions 2 and 3 correspond to cases which are not obtained as direct sums of lower dimensional ones. Simple eigenvalues are not listed.

Next we describe versal deformations of $\mathrm{o}(p, q)$ elements belonging to strata of codimension 1 and those belonging to codimension 2 which are not intersections of two strata of codimension 1 (cf column $c = 1$ and the underlined entries of column $c = 2$ of table 8). The normal forms M_J and their versal deformations are given in table 9 for the smallest value of $p + q$ in which they occur. For any greater $p + q$ the corresponding matrix of table 9 has to be enlarged with a diagonal matrix with distinct eigenvalues, depending on additional parameters μ_i. When all the parameters λ_i and μ_j are put equal to zero, one has the normal form M_J. Parameters λ are used in the description of both orbits and strata, while when changing a μ_i one stays within the same stratum. The number of parameters μ_i is subtracted from the codimension d of an orbit in order to get

Table 7. Jordan normal forms of elements of $o(p, q)$, $p+q \leq 6$. Details of notation in § 4.

o(2)	d	c
$\pm ib$	1	0

o(1, 1)	d	c
$\pm a$	1	0

o(3)	d	c
$(\pm ib)\,0$	1	0
$0\,0\,0$	3	3

o(2, 1)	d	c
$\pm(ib)\,0$	1	0
$(\pm a)\,0$	1	0
0^3	1	1
$0\,0\,0$	3	3

o(3, 1)	d	c
$(\pm a)(\pm ib)$	2	0
$0\,0\,0\,0$	6	6
$0\,0^3$	2	2

o(4, 0)	d	c
$(\pm ib)(\pm ib')$	2	0
$(\pm ib)(\pm ib)$	4	3
$0\,0\,0\,0$	6	6

o(2, 2)	d	c
$(+a)(+a')$	2	0
$(+a)(+a)$	4	3
$(+a)^2$	2	1
$(+a+ib)$	2	0
$(+ib)^2_+$	2	1
$(\pm ib)^2_-$	2	1
$0\,0\,0\,0$	6	6
$(+0)^2$	4	4
$(+ib)(+ib')$	2	0
$(+ib)(+ib)$	4	3
$0^3_+\,0_-$	2	2
$0^3_-\,0_+$	2	2

o(4, 1)	d	c
$0^3\,(\pm ib)$	2	1
$0\,(\pm a)(\pm ib)$	2	0
$0\,0\,0\,(\pm a)$	4	3
$0\,0\,0\,(\pm ib)$	4	3
$0\,0\,0\,0\,0$	10	10
$0^3\,0\,0$	4	4
$(\pm ib)(\pm ib')\,0$	2	0
$(\pm ib)(\pm ib)\,0$	4	3

o(3, 2)	d	c
$0^3\,(\pm ib)$	2	1
$0\,(\pm a)(\pm ib)$	2	0
$0\,0\,0\,(\pm a)$	4	3
$0_+\,0_+\,0_+\,(\pm ib)_-$	4	3
$0_+\,0_-\,0_-\,(\pm ib)_+$	4	3
$0\,0\,0\,0\,0$	10	10
$0^3_+\,0_+\,0_-$	4	4
$0^3_-\,0_+\,0_+$	4	4
$(\pm ib)(\pm ib')\,0$	2	0
$(\pm ib)(\pm ib)\,0$	4	3
$0\,(\pm a)(\pm a')$	2	0
$0\,(\pm a)(\pm a)$	3	2
$0\,(\pm a)^2$	2	1
$0\,(\pm ib)^2_+$	2	1
$0\,(\pm ib)^2_-$	2	1
$0\,(\pm a \pm ib)$	2	0
$0\,(\pm 0)^2$	6	6
$0^3\,(\pm a)$	2	1
0^5	2	2

o(5, 0)	d	c
$0\,0\,0\,0\,0$	10	10
$0\,0\,0\,(\pm ib)$	4	3
$0\,(\pm ib)(\pm ib')$	2	0
$0\,(\pm ib)(\pm ib)$	4	3

o(4, 2)	d	c
$(\pm a)(\pm a')(\pm ib)$	3	0
$(\pm a)^2(\pm ib)$	3	1
$0\,0\,0\,0\,(\pm a)$	7	6
$0_+\,0_+\,0_+\,0_+\,(\pm ib)_-$	7	6
$0_+\,0_+\,0_-\,0_-\,(\pm ib)_+$	7	6
$0\,0\,0\,0\,0\,0$	15	15
$0\,0\,(\pm 0)^2$	9	9
$0^3\,0\,(\pm a)$	3	2
$(\pm 0)^2\,(\pm ib)$	5	4
$0^3_+\,0_+\,0_+\,0_-$	7	7
$0^3_-\,0_+\,0_+\,0_+$	7	7
$0^3\,0^3$	5	5
$0^5\,0$	3	3
$(\pm a \pm ib)(\pm ib')$	3	0
$(\pm ib)^2_+(\pm ib')_+$	3	1
$(\pm ib)^2_+(\pm ib)_+$	5	4
$(\pm ib)^2_-(\pm ib')_+$	3	1
$(\pm ib)^2_-(\pm ib)_+$	5	4
$(\pm ib)^3$	3	2
$(\pm ib)(\pm a)(\pm a)$	5	3
$(\pm ib)(\pm ib')(\pm ib'')$	3	0
$(\pm ib)_+(\pm ib)_+(\pm ib')_-$	5	3
$(\pm ib)_+(\pm ib)_-(\pm ib')_+$	5	3
$(\pm ib)(\pm ib)(\pm ib)$	9	8
$(\pm ib)_+\,0^3_+\,0_-$	3	2
$(\pm ib)_+\,0^3_-\,0_+$	3	2

o(3, 3)	d	c
$(\pm a)(\pm a')(\pm a'')$	3	0
$(\pm a)(\pm a)(\pm a')$	5	3
$(\pm a)^2(\pm a')$	3	1
$(\pm ib)(\pm ib')(\pm a)$	3	0
$(\pm ib)(\pm ib)(\pm a)$	5	3
$(\pm ib)^2_+(\pm a)$	3	1
$(\pm ib)^2_-(\pm a)$	3	1
$(\pm a)(\pm a)(\pm a)$	9	8
$(\pm a)^2(\pm a)$	5	4
$(\pm a)^3$	3	2
$(\pm a \pm ib)(\pm a')$	3	0
$(\pm 0)^2\,(\pm a)$	5	4
$0\,0\,0\,0\,(\pm a)$	7	6
$(\pm 0)^2\,0\,0$	9	9
$0\,0\,0\,0\,0\,0$	15	15
$(\pm ib)_+\,0_+\,0_-\,0_-\,0_-$	7	6
$(\pm ib)_-\,0_-\,0_+\,0_+\,0_+$	7	6
$0^5_-\,0_+$	3	3
$0^5_+\,0_-$	3	3
$0^3\,(\pm a)\,0$	3	2
$0^3\,0\,0\,0$	7	7
$(\pm ib)_+\,0^3_-\,0_-$	3	2
$(\pm ib)_-\,0^3_+\,0_+$	3	2
$0^3\,0^3$	5	5

o(5, 11)	d	c
$0\,0\,0\,0\,(\pm a)$	7	6
$0\,0\,0\,0\,0\,0$	15	15
$0\,0\,0\,0^3$	7	7
$(\pm a)(\pm ib)(\pm ib')$	3	0
$(\pm a)(\pm ib)(\pm ib)$	5	3
$0\,0\,0\,0\,(\pm ib)$	7	6
$0\,0^3\,(\pm ib)$	3	2

o(6, 0)	d	c
$0\,0\,0\,0\,0\,0$	15	15
$0\,0\,0\,0\,(\pm ib)$	7	6
$(\pm ib)(\pm ib')(\pm ib'')$	3	0
$(\pm ib)(\pm ib)(\pm ib')$	5	3
$(\pm ib)(\pm ib)(\pm ib)$	9	8

Table 8. Strata in $o(p, q)$ of codimension $c \leq 3$. Strata which are not obtained as intersections of strata of smaller codimension are underlined. Details of notation in § 4.

$c = 1$	$c = 2$	$c = 3$
0^3 $(\pm a)^2$ $(\pm ib)^2$	$0^3(\pm a)^2$, $0^3(\pm ib)^2$ $(\pm a)^2(\pm ib)^2$, $\underline{(\pm a)^3}$ $\underline{(\pm ib)^3}$, $\underline{(\pm a \pm ib)^2}$ $\underline{0^5}$, $\underline{0^3 0}$, $(\pm a)^2(\pm a')^2$ $(\pm ib)^2(\pm ib')^2$	$(\pm a)^3 0^3$, $(\pm ib)^3 0^3$, $(\pm a)^2(\pm a')^2 0^3$, $(\pm a)^2(\pm ib)^2 0^3$, $(\pm ib)^2(\pm ib')^2 0^3$, $(\pm a)^3(\pm a')^2$, $(\pm a)^3(\pm ib)^2$, $(\pm ib)^3(\pm a)^2$, $(\pm ib)^3(\pm ib')^2$, $(\pm a)^2 0^5$, $(\pm ib)^5 0^5$, $(\pm a)^2 0^3 0$, $(\pm ib)^2 0^3 0$, $(\pm a)^2(\pm a')^2(\pm a'')^2$, $(\pm a)^2(\pm a')^2(\pm ib)^2$, $(\pm a)^2(\pm ib)^2(\pm ib')^2$, $(\pm ib)^2(\pm ib')^2(\pm ib'')^2$, $\underline{0^7}$, $\underline{0^5 0}$, $\underline{(\pm a)^4}$, $(\pm ib)^4$, $\underline{(\pm a)(\pm a)}$, $\underline{(\pm ib)(\pm ib)}$, $(\pm a \pm ib)^2 0^3$, $(\pm a \pm ib)^2(\pm a')$, $(\pm a \pm ib)^2(\pm ib')$

the codimension c of a stratum. If only representatives of strata are desired, all parameters μ_i of table 9 should be put equal to zero.

In table 10 we give the bifurcation diagrams corresponding to all examples appearing in table 9. The bifurcation diagrams are obtained in the following way: one takes the versal deformation in table 9, equates the μ_i to zero and then asks that the characteristic polynomial of the resulting matrix has a zero discriminant. We give the equation of the discriminant, draw the curve and give in each region of the plane the respective position of the eigenvalues. Table 10 contains all cases of codimensions 1 and 2 which are not an intersection of two cases of codimension 1 (column $c = 1$ and underlined entries of column $c = 2$ of table 8). We give also an example of one case of codimension 2 which is the direct sum of two cases of codimension 1, namely $(\pm a)^2(\pm ib)^2$.

Finally let us describe the translation of a versal deformation described in this article into a fixed 'physical basis'. We chose the Lie algebra o(4, 2) of the conformal group of space–time and a basis common in physics (cf appendix). Suppose we want to find versal deformations in o(4, 2) of the element K_2, where K_2 is one of the three generators of Lorentz boosts.

In the basis (A3), $K_2 = \Omega_{35}$ (cf A1). We bring it to a diagonal form using the matrix

$$g = \frac{1}{2}\begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{2} & \sqrt{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & +\sqrt{2} & -\sqrt{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \end{pmatrix}. \tag{4.3}$$

More precisely,

$$g^{-1}K_2 g = M_J = \begin{pmatrix} 0 & & & & & \\ & 0 & & & & \\ & & 1 & & & \\ & & & -1 & & \\ & & & & 0 & \\ & & & & & 0 \end{pmatrix} \tag{4.4}$$

Table 9. Versal deformations of representatives of strata of codimension 1 and 2 in o(p, q), which are not intersections of strata of smaller codimension.

c = 1

0^3

$$\begin{pmatrix} 0 & 1 & 0 \\ \lambda_1 & 0 & 1 \\ 0 & \lambda_1 & 0 \end{pmatrix}$$

$(\pm a)^2$

$$\begin{pmatrix} a & 1 & 0 & 0 \\ \lambda_1 & a+\mu_1 & 0 & 0 \\ 0 & 0 & -a & -\lambda_1 \\ 0 & 0 & -1 & -a-\mu_1 \end{pmatrix}$$

$(\pm ib)^2$

$$\begin{pmatrix} 0 & b & 1 & 0 \\ -b+\mu_1 & 0 & 0 & 1 \\ \lambda_1 & 0 & 0 & b \\ 0 & \lambda_1 & -b+\mu_1 & 0 \end{pmatrix}$$

c = 2

$(\pm a)^3$

$$\begin{pmatrix} a & 1 & 0 & 0 & 0 & 0 \\ 0 & a & 1 & 0 & 0 & 0 \\ \lambda_2 & \lambda_1 & a+\mu_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -a & 0 & -\lambda_2 \\ 0 & 0 & 0 & -1 & -a & -\lambda_1 \\ 0 & 0 & 0 & 0 & -1 & -a+\mu_1 \end{pmatrix}$$

$(+a+ib)^2$

$$\begin{pmatrix} a & b & 1 & 0 & 0 & 0 & 0 & 0 \\ -b & a & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & a & b & 0 & 0 & 0 & 0 \\ \lambda_2 & \lambda_1 & -b+\mu_2 & a+\mu_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -a & b & 0 & -\lambda_2 \\ 0 & 0 & 0 & 0 & -b & -a & 0 & -\lambda_1 \\ 0 & 0 & 0 & 0 & -1 & 0 & -a & b-\mu_2 \\ 0 & 0 & 0 & 0 & 0 & -1 & -b & -a-\mu_1 \end{pmatrix}$$

$(\pm ib)^3$

$$\begin{pmatrix} 0 & b-\mu_1 & 1 & 0 & 0 & 0 \\ -b & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & b & 1 & 0 \\ 0 & \lambda_1 & -b & 0 & 0 & 1 \\ 0 & -\lambda_2 & 0 & 0 & 0 & b \\ \lambda_2 & 0 & 0 & \lambda_1 & -b+\mu_1 & 0 \end{pmatrix}$$

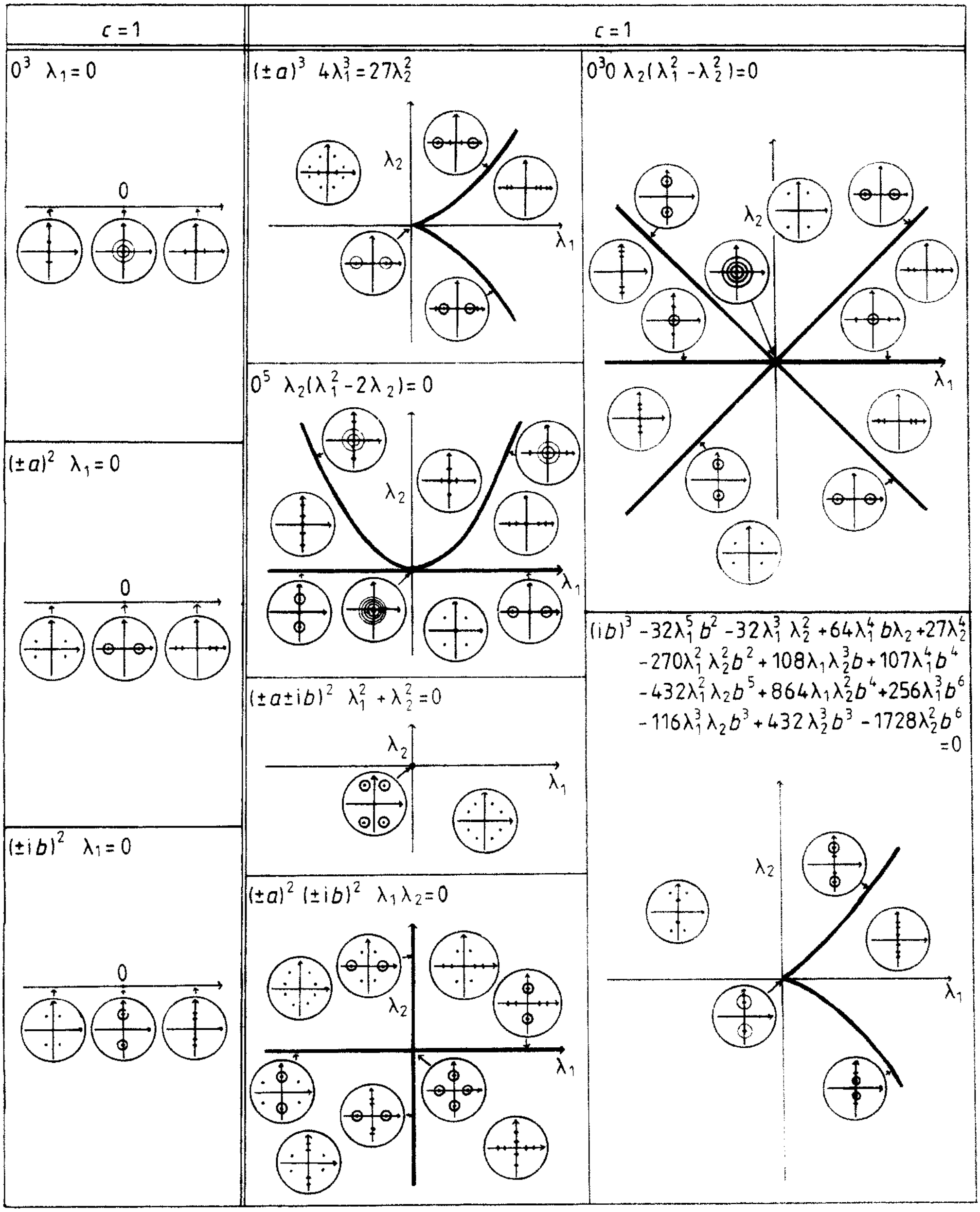
0^5

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ \lambda_1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \lambda_2 & 0 & 0 & 0 & 1 \\ 0 & \lambda_2 & 0 & \lambda_1 & 0 \end{pmatrix}$$

$0^3 0$

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ \lambda_1 & 0 & 1 & 0 \\ 0 & \lambda_1 & 0 & \lambda_2 \\ -\lambda_2 & 0 & 0 & 0 \end{pmatrix}$$

Table 10. Bifurcation diagrams of 1- and 2-parameter families of elements of o(p, q) in general position. The different circles indicate the position of the eigenvalues in each region of the (λ_1–λ_2) plane. The interiors of the circles are portions of the complex plane, dots and concentric circles indicating multiple eigenvalues with the corresponding multiplicity.



The matrix verifying $M_J K' + K' M_J^T = 0$ is

$$K' = \begin{pmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 0 & 1 & & \\ & & 1 & 0 & & \\ & & & & -1 & \\ & & & & & 1 \end{pmatrix} = g^T \begin{pmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & -1 & \\ & & & & & -1 \end{pmatrix} g. \quad (4.5)$$

Diagonalising K', one verifies that it has the correct signature, namely $(1\ 1\ 1\ -1\ -1\ 1)$.

We next find the versal deformation of M_J. Since $c = 3$ it does not appear in table 9. From the general results in table 3 it follows that the deformation is given by

$$M_A(\lambda, \mu) = \begin{pmatrix} 0 & \lambda_1 & 0 & 0 & \lambda_3 & \lambda_4 \\ -\lambda_1 & 0 & 0 & 0 & \lambda_5 & \lambda_6 \\ 0 & 0 & 1+\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & -1-\mu & 0 & 0 \\ \lambda_3 & \lambda_5 & 0 & 0 & 0 & \lambda_2 \\ -\lambda_4 & -\lambda_6 & 0 & 0 & \lambda_2 & 0 \end{pmatrix} \quad (4.6)$$

where $\lambda_1, \ldots, \lambda_6, \mu \in \mathbb{R}$ and $M_A(0, 0) = M_J$. (M_J was of type I+I+III+I′+I.)

In order to identify the deformation in terms of the physical generators of the appendix, $M_A(\lambda, \mu)$ has to be brought back in the basis (A3). We obtain

$$M'_A(\lambda, \mu) = g M_A(\lambda, \mu) g^{-1} = \begin{pmatrix} 0 & \lambda_1 & 0 & \lambda_4 & 0 & \lambda_3 \\ -\lambda_1 & 0 & 0 & \lambda_6 & 0 & \lambda_5 \\ 0 & 0 & 0 & 0 & 1+\mu & 0 \\ -\lambda_4 & -\lambda_6 & 0 & 0 & 0 & \lambda_2 \\ 0 & 0 & 1+\mu & 0 & 0 & 0 \\ \lambda_3 & \lambda_5 & 0 & \lambda_2 & 0 & 0 \end{pmatrix}. \quad (4.7)$$

The versal deformation $M'_A(\lambda, \mu)$ of the generator K_2 is now in the basis (A1). Therefore, by an obvious change of parameters it can be written as

$$M'_A(\lambda, \mu) = K_2 + \mu K_2 + \lambda'_1 P_1 + \lambda'_2 C_1 + \lambda'_3 P_3 + \lambda'_4 C_3 + \lambda'_5 L_2 + \lambda'_6 D. \quad (4.8)$$

Putting $\mu = 0$ (4.8) gives representatives of all strata in the neighbourhood of K_2. The Jordan normal form of K_2 contains no non-zero off-diagonal elements, and therefore also the versal deformation (4.8) is relatively obvious; it turns out to be a linear combination of o(4, 2) generators which commute with K_2.

Appendix

In physics the usual basis of the Lie algebra o(4, 2) of the conformal group of space–time consists of the following generators: rotations L_i, Lorentz boosts K_i, dilation D, translations P_μ, and special conformal transformations C_μ, where $i = 1, 2, 3$ and $\mu = 0, 1, 2, 3$. They satisfy the standard commutation relations (cf for instance, Burdet *et al*

1978). As 6×6 real matrices the generators can be chosen in the following way

$$\begin{array}{lll} L_1=\Omega_{23} & L_2=-\Omega_{24} & L_3=\Omega_{34} \\ K_1=\Omega_{45} & K_2=\Omega_{35} & K_3=\Omega_{25} \\ P_1-C_1=2\Omega_{14} & P_2-C_2=2\Omega_{13} & P_3-C_3=2\Omega_{12} \\ P_1+C_1=2\Omega_{46} & P_2+C_2=2\Omega_{36} & P_3+C_3=2\Omega_{26} \\ P_0-C_0=-2\Omega_{15} & P_0+C_0=2\Omega_{56} & D=-\Omega_{16} \end{array} \tag{A1}$$

where

$$\Omega_{ij}=\begin{cases} E_{ij}-E_{ji} & \text{for } 1\leqslant i,j\leqslant 4 \text{ and } 5\leqslant i,j\leqslant 6 \\ E_{ij}+E_{ji} & \text{otherwise.} \end{cases} \tag{A2}$$

Here E_{ij} is the 6×6 matrix with 1 at the intersection of the ith row and the jth column. Any matrix X of (A1) belongs to the o(4, 2) algebra because it verifies the defining relation $XK+KX^{\mathrm{T}}=0$ with

$$K=\begin{pmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & -1 & \\ & & & & & -1 \end{pmatrix}. \tag{A3}$$

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